

# Noncommutative superspace and super Heisenberg group

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**ABSTRACT:** In this paper, we consider noncommutative superspace in relation with super Heisenberg group. We construct a matrix representation of super Heisenberg group and apply this to the two-dimensional deformed  $\mathcal{N} = (2, 2)$  superspace that appeared in string theory. We also construct a toy model for non-centrally extended ‘super Heisenberg group’.

**KEYWORDS:** Superspaces, Non-Commutative Geometry.

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**1. Introduction**

Noncommutative geometry [1] naturally appears in string theory [2, 3]. Low-energy effective theory of D-branes in a background NSNS  $B$ -field becomes the noncommutative field theory where the spacetime coordinates  $x^\mu$  are noncommutative  $[x^\mu, x^\nu] \neq 0$  [4–6]. On the other hand, when we turn on the background RR field, low-energy effective theory of D-branes becomes the field theory on non(anti)commutative superspace of which the fermionic coordinate  $\theta^\alpha$  has nontrivial commutation relation  $\{\theta^\alpha, \theta^\beta\} \neq 0$  [7–13]. Gauge theories on non(anti)commutative superspace are studied extensively [14–17]. In the case of constant  $B$ -field, the algebra of the coordinates becomes  $[x^\mu, x^\nu] = \text{const.}$ , which is called Heisenberg algebra (or Weyl algebra as an universal enveloping algebra of Heisenberg algebra). Heisenberg group [18–20] is constructed from the Heisenberg algebra by exponential mapping. If we include both of background NSNS and RR fields, it is expected that super Heisenberg group [21] would appear.

Heisenberg group and its Schrödinger representation are defined rigourously in mathematics from the motivation of quantum mechanics [20]. Heisenberg group is constructed as a central extension of a symplectic vector space and its matrix representation is constructed by triangular matrices. The commutation relation  $[x^\mu, x^\nu] = \text{const.}$  is regarded as the operator representation (Schrödinger representation) of Heisenberg group. The matrix representation of Heisenberg group is useful to construct noncommutative tori [22–24] and quantum theta-functions [25–30]. Here noncommutative tori are defined by the commutator relation coming from the cocycle condition used in Heisenberg group. In this paper, for the analogue of bosonic Heisenberg group, we find the representation of super Heisenberg



The algebra of the functions on noncommutative space is equivalent to the algebra of the functions on commutative space with the noncommutative product, which is called Moyal product:

$$F(X) * G(X) = \exp\left(\frac{i}{2}\Theta^{\mu\nu} \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X'^\nu}\right) F(X)G(X') \Big|_{X'^\mu=X^\mu}. \quad (2.5)$$

## 2.2 Heisenberg group

We define a Heisenberg group,  $Heis(\mathbb{R}^{2n}, \psi)$ . As a set  $Heis(\mathbb{R}^{2n}, \psi)$  is  $U(1) \times \mathbb{R}^{2n}$ . For  $t, t' \in U(1)$ , and  $(x, y), (x', y') \in \mathbb{R}^{2n}$ , we define  $(t, x, y), (t', x', y') \in Heis(\mathbb{R}^{2n}, \psi)$ ,

$$(t, x, y) \cdot (t', x', y') = (t + t' + \psi(x, y; x', y'), x + x', y + y'), \quad (2.6)$$

where  $\psi : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ , satisfies the cocycle condition

$$\psi(x, y; x', y')\psi(x + x', y + y'; x'', y'') = \psi(x, y; x' + x'', y' + y'')\psi(x', y'; x'', y''), \quad (2.7)$$

which is a necessary and sufficient condition for the multiplication to be associative. There is an exact sequence

$$0 \rightarrow \mathbb{R} \xrightarrow{i} Heis(\mathbb{R}^{2n}, \psi) \xrightarrow{j} \mathbb{R}^{2n} \rightarrow 0 \quad (2.8)$$

called a central extension, with the inclusion  $i(t) = (t, 0, 0)$  and the projection  $j(t, x, y) = (x, y)$ , where  $i(\mathbb{R})$  is the center in  $Heis(\mathbb{R}^{2n}, \psi)$ .

We now introduce two representations of this Heisenberg group  $Heis(\mathbb{R}^{2n}, \psi)$ . One is a matrix representation and the other is an operator representation.

First we introduce the matrix representation  $Heis(\mathbb{R}^{2n}, \psi) \rightarrow Mat_{(n+2, n+2)}(\mathbb{R})$  in two ways. One of them is given by

$$(t, x, y) \rightarrow \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{M}(t, x, y). \quad (2.9)$$

In this case  $\psi(x, y; x', y') = xy'$ .

For the other one, we use the Lie algebra and Lie group approach. Let

$$M(t, x, y) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.10)$$

Then

$$\mathcal{M}(t, x, y) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = I + M(t, x, y). \quad (2.11)$$

For the second matrix representation, we define

$$\pi(t, x, y) := e^{M(t, x, y)} = I + M + M^2/2 + \dots = \begin{pmatrix} 1 & x & t + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.12)$$

Then,  $Heis(\mathbb{R}^{2n}, \psi)$  with the cocycle  $\psi(x, y; x', y') = \frac{1}{2}(xy' - yx')$  is isomorphically embedded in  $Mat_{(n+2, n+2)}(\mathbb{R})$ . In this case,  $\psi$  becomes a symplectic form, and then this representation can be extended to general symplectic vector spaces.<sup>3</sup> We can easily show that both  $\mathcal{M}$  in (2.9) and  $\pi$  in (2.12) are group homomorphisms, which shows that they are matrix representations of the Heisenberg group.

The Lie algebra introduced in the second matrix representation is a vector space isomorphic to  $\mathbb{R} \times \mathbb{R}^{2n}$  generated by  $q_i, p_i$  ( $i = 1, \dots, n$ ), and  $r$  such that

$$[q_i, p_j] = \delta_{ij}r, \quad \text{others} = 0. \tag{2.13}$$

The above Lie algebra is the so-called Heisenberg algebra,  $h(2n)$ . A general element of this Heisenberg algebra  $h(2n)$  is of the form  $x \cdot q + y \cdot p + tr$ . In the second matrix representation, the generators are mapped as follows

$$\begin{aligned} q_i \rightarrow Q_i &= \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}, & p_i \rightarrow P_i &= \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & 1 \\ & & & & \vdots \\ & & & & 0 \end{pmatrix}, \\ r \rightarrow R &= \begin{pmatrix} 0 & \dots & \dots & \dots & 1 \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}, \end{aligned} \tag{2.14}$$

where the component 1 in  $Q_i$  appears only in the  $(i + 1)$ -th column of the first row, and the component 1 in  $P_i$  appears only in the  $(i + 1)$ -th row of the last column. In this notation,  $\pi(t, x, y)$  in (2.12) can be expressed as

$$\pi(t, x, y) = e^{x \cdot Q + y \cdot P + tR}. \tag{2.15}$$

Note that  $\pi(t, x, y) \cdot \pi(t', x', y')$  can be expressed as

$$e^{M(t, x, y)} \cdot e^{M(t', x', y')} = e^{M(t, x, y) + M(t', x', y') + \frac{1}{2}[M(t, x, y), M(t', x', y')] + \dots} \tag{2.16}$$

and we will use this property in our computations.

Next, we explain the Schrödinger representation which is an operator representation on  $L^2(\mathbb{R}^n)$  of the Heisenberg group, where  $L^2(\mathbb{R}^n)$  is the completion of the Schwarz space on  $\mathbb{R}^n$ . For this we reexpress the commutation relation of the Heisenberg algebra (2.13) with new generators  $X_i, Y_i$ , and  $I$  by the following map

$$q_i \longrightarrow iX_i, \quad p_i \longrightarrow iY_i, \quad r \longrightarrow -iI. \tag{2.17}$$

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<sup>3</sup>In general,  $(x, y)$  is a Darboux pair of the symplectic vector space and  $\psi$  is the symplectic form.

Then we have

$$[X_i, Y_j] = i\delta_{ij}I, \quad \text{others} = 0. \quad (2.18)$$

A representation for  $X_i, Y_i, I$  as operators can be given by

$$(X_i f)(s) = s_i f(s), \quad (Y_i f)(s) = -i \frac{\partial f}{\partial s_i}(s), \quad I = \text{identity}, \quad (2.19)$$

where  $s = (s_1, \dots, s_n)$ . Note that the above  $X_i, Y_i$  can be regarded as the noncommutative coordinates  $X^\mu$  introduced in the previous subsection.

For the operator representation of the first matrix representation  $\mathcal{M}(t, x, y)$  with the cocycle  $\psi(x, y; x', y') = xy'$ , we send  $(t, x, y) \rightarrow \chi(t)e(x)d(y)$ , where

$$\begin{aligned} \chi(t) : f(s) &\rightarrow e^{-it} f(s), \\ e(x) : f(s) &\rightarrow e^{ix \cdot X} f(s) = e^{ix \cdot s} f(s), \\ d(x) : f(s) &\rightarrow e^{iy \cdot Y} f(s) = f(s + y). \end{aligned} \quad (2.20)$$

For the operator representation of the second matrix representation  $\pi(t, x, y)$  with the cocycle  $\psi(x, y; x', y') = \frac{1}{2}(xy' - yx')$ , we send  $(t, x, y)$  to  $U(t, x, y) = \chi(t + \frac{xy}{2})e(x)d(y)$ . Note that  $U(t, x, y)$  can be rewritten as

$$U(t, x, y) = \exp[i(x \cdot X + y \cdot Y - tI)], \quad (2.21)$$

and this corresponds to  $\pi(t, x, y)$  in (2.15) via (2.17)

The noncommutative parameter is

$$\Theta(x, y; x', y') = \psi(x, y; x', y') - \psi(x', y'; x, y) = xy' - yx' \quad (2.22)$$

for both cases. In the second case,  $\psi(x, y; x', y') = \frac{1}{2}(xy' - yx')$ , so that  $\Theta = 2\psi$ .

In the later part of our work, we will use the second operator representation, which can be mapped into a noncommutative space as we mentioned above, is irreducible and unitary. And due to Stone-von Neumann-Mackey theorem, this is unique up to isomorphism.

Now we explain Stone-von Neumann-Mackey theorem [18–20]. First we define  $e : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$  by  $e(z_1, z_2) = \tilde{z}_1 \tilde{z}_2 \tilde{z}_1^{-1} \tilde{z}_2^{-1}$  where  $z_1, z_2 \in \mathbb{R}^{2n}$ , and  $\tilde{z}_i$  is a lifting of  $z_i$  which means  $j(\tilde{z}_i) = z_i$ . Then  $e$  is well defined independent of the choice of  $\tilde{z}_i$  and is a skew-symmetric pairing. In our case,  $e$  corresponds to  $\Theta$  in (2.22). If for some subgroup  $H \subset \mathbb{R}^{2n}$ ,  $e|_{H \times H} = 0$ , then  $H$  is called an isotropic subspace (or Lagrangian). If  $H$  is maximal among those isotropic subspaces, it is called a maximal isotropic subspace. For example,  $H$  can be the  $x$ -space or the  $y$ -space or some combination. Now we state the theorem [18].

**Theorem 1.** (Stone, von Neumann, Mackey)

Let  $G = \text{Heis}(\mathbb{R}^{2n}, \psi)$  be a Heisenberg group. Then

(1)  $G$  has a unique irreducible unitary representation

$$U : G \rightarrow \text{Aut}(\mathcal{H}_0) \quad (2.23)$$

such that  $U_t = e^{-it} \cdot I$  for all  $t \in \mathbb{R}$ .

(2) For all maximal isotropic subgroup  $H \subset \mathbb{R}^{2n}$ , and a lifting  $\sigma(z) = (\alpha(z), z)$  of  $j$  over  $H$ , where  $\alpha$  is a group homomorphism from  $H$  to  $\mathbb{R}$ , this representation may be realized by

$\mathcal{H}_0 = \{ \text{measurable function } f : \mathbb{R}^{2n} \rightarrow \mathbb{C}, \text{ such that}$

$$\begin{aligned} a) \quad & f(z+h) = \alpha(h)^{-1} \psi(h, z)^{-1} f(z), \quad \forall h \in H, \\ b) \quad & \int |f(z)|^2 dz < \infty. \quad U_{(t, z')} f(z) = e^{-it} \psi(z, z') f(z+z'), \quad \forall z' \in \mathbb{R}^{2n}. \end{aligned}$$

(3) All representations  $(U, \mathcal{H})$  such that  $U_t = e^{-it} \cdot I$ , all  $t \in \mathbb{R}$ , are isomorphic to  $\mathcal{H}_0 \otimes \mathcal{H}_1$ , and  $G$  acting trivially on  $\mathcal{H}_1$ .

### 3. Super Heisenberg group

Now, we consider the extension of the work in the previous section to the super case. As in the bosonic case, we define the super Heisenberg group,  $sHeis(\mathbb{R}^{2n|2m}, \psi)$ , as follows. For  $t, t' \in \mathbb{U}(1)$ , and  $(x, \alpha), (y, \beta), (x', \alpha'), (y', \beta') \in \mathbb{R}^{n|m}$ , we define  $(t, x, y, \alpha, \beta), (t', x', y', \alpha', \beta') \in sHeis(\mathbb{R}^{2n|2m}, \psi)$  such that

$$(t, x, y, \alpha, \beta) \cdot (t', x', y', \alpha', \beta') = (t+t' + \psi(x, y, \alpha, \beta; x', y', \alpha', \beta'), x+x', y+y', \alpha+\alpha', \beta+\beta'), \quad (3.1)$$

where  $\psi : \mathbb{R}^{2n|2m} \times \mathbb{R}^{2n|2m} \rightarrow \mathbb{R}$ , satisfies the cocycle condition

$$\begin{aligned} & \psi(x, y, \alpha, \beta; x', y', \alpha', \beta') \psi(x+x', y+y', \alpha+\alpha', \beta+\beta'; x'', y'', \alpha'', \beta'') \\ & = \psi(x, y, \alpha, \beta; x'+x'', y'+y'', \alpha'+\alpha'', \beta'+\beta'') \psi(x', y', \alpha', \beta'; x'', y'', \alpha'', \beta''), \end{aligned} \quad (3.2)$$

a necessary and sufficient condition for associative multiplication. Now, there is an exact sequence

$$0 \rightarrow \mathbb{R} \xrightarrow{i} sHeis(\mathbb{R}^{2n|2m}, \psi) \xrightarrow{j} \mathbb{R}^{2n|2m} \rightarrow 0, \quad (3.3)$$

a central extension, with the inclusion  $i(t) = (t, 0)$ , the projection  $j(t, z) = z$ ,  $z \in \mathbb{R}^{2n|2m}$ , where  $i(\mathbb{R})$  is the center in  $sHeis(\mathbb{R}^{2n|2m}, \psi) := sH(2n|2m)$ . As in the bosonic case, we can introduce two types of matrix representations and the corresponding operator representations for the super Heisenberg group.

First, we consider the matrix representations,  $sH(2n|2m) \rightarrow Mat_{(n+2|m, n+2|m)}(\mathbb{R})$ . The first matrix representation is given by

$$(t, x, y, \alpha, \beta) \rightarrow \left( \begin{array}{ccc|c} 1 & x & t & \alpha \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & \beta & 1 \end{array} \right) = \mathcal{M}(t, x, y, \alpha, \beta). \quad (3.4)$$

In this case  $\psi(x, y, \alpha, \beta; x', y', \alpha', \beta') = xy' + \alpha\beta'$ .

For the Lie algebra and Lie group approach, we let

$$\mathcal{M}(t, x, y, \alpha, \beta) := I + M(t, x, y, \alpha, \beta), \quad (3.5)$$

where

$$M(t, x, y, \alpha, \beta) = \left( \begin{array}{ccc|c} 0 & x & t & \alpha \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \beta & 0 \end{array} \right). \quad (3.6)$$

We then define

$$\pi(t, x, y, \alpha, \beta) := e^{M(t, x, y, \alpha, \beta)} = I + M + M^2/2 + \dots = \left( \begin{array}{ccc|c} 1 & x & t + \frac{xy + \alpha\beta}{2} & \alpha \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & \beta & 1 \end{array} \right). \quad (3.7)$$

In this case  $\psi(x, y, \alpha, \beta; x', y', \alpha', \beta') = \frac{1}{2}(xy' - yx' + \alpha\beta' + \beta\alpha')$ .

The super Lie algebra introduced in the second matrix representation is a vector space isomorphic to  $\mathbb{R} \times \mathbb{R}^{2n|2m}$  generated by  $q_i, p_i, \xi_a, \lambda_a$  ( $i = 1, \dots, n, a = 1, \dots, m$ ), and  $r$  such that

$$[q_i, p_j] = \delta_{ij}r, \quad \{\xi_a, \lambda_b\} = \delta_{ab}r, \quad \text{others} = 0. \quad (3.8)$$

The above super Lie algebra is the so-called super Heisenberg algebra,  $sh(2n|2m)$ . A general element of this super Heisenberg algebra  $sh(2n|2m)$  is of the form  $x \cdot q + y \cdot p + \alpha \cdot \xi + \beta \cdot \lambda + tr$ . In this representation, the generators are mapped as follows similar to the bosonic case:

$$\begin{aligned} q_i &\rightarrow \left( \begin{array}{ccc|c} 0 & e_i^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), & p_i &\rightarrow \left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), & r &\rightarrow \left( \begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), \\ \xi_a &\rightarrow \left( \begin{array}{ccc|c} 0 & 0 & 0 & e_a^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), & \lambda_a &\rightarrow \left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e_a & 0 \end{array} \right), \end{aligned} \quad (3.9)$$

where  $e_i$  is the column vector in which the  $i$ -th component is 1 and the others vanish, and the same for  $e_a$ .

Let  $V$  be a real super vector space of dimension  $2n|2m$  with non-degenerate skew-symmetric form  $(, )$ . Let  $q_i, p_i, \xi_a, \lambda_a$ , for  $i = 1, \dots, n$  and  $a = 1, \dots, m$ , be a basis of  $V$  such that the matrix of  $(, )$  with respect to this basis is

$$\phi = \begin{pmatrix} 0 & 1 & \mathbf{0} \\ -1 & 0 & \\ & 0 & 1 \\ \mathbf{0} & 1 & 0 \end{pmatrix} \quad (3.10)$$

i.e.  $(q_i, q_j) = (p_i, p_j) = (\xi_a, \xi_b) = (\lambda_a, \lambda_b) = 0$  and  $(q_i, p_j) = \delta_{ij}$ ,  $(\xi_a, \lambda_b) = \delta_{ab}$ . Here  $\phi$  corresponds to the map  $e$  which we introduced for Stone-von Neumann-Mackey theorem in the bosonic case. So  $V = V_{\bar{0}} \otimes V_{\bar{1}}$ , where  $\{q_i, p_i\}$  is a basis for  $V_{\bar{0}}$  and  $\{\xi_a, \lambda_a\}$  a basis



for  $V_{\bar{1}}$ . The super Heisenberg algebra  $sh(V)$  can be constructed as a central extension of the Abelian Lie superalgebra  $V$  by an even generator  $r$  [21]. We have an exact sequence  $0 \rightarrow \mathbb{R} \cdot r \rightarrow sh(V) \rightarrow V \rightarrow 0$ , and the Lie bracket is defined by  $[u, v] = (u, v)r, \forall u, v \in V$ . Then,  $sh(V) = sh(2n|2m)$ .

The operator representation can be given as in the bosonic case. The target space for the super Schrödinger representation is  $L^2(\mathbb{R}^{n|m}) := L^2(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^m)^*$ , which is the completion of the Schwarz space  $S(\mathbb{R}^{n|m}) := S(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^m)^*$ . Here  $\Lambda^*(\mathbb{R}^m)^*$  is the vector space spanned by  $\{v_1 \wedge \cdots \wedge v_l | v_i = (1, \dots, l) \in \mathbb{R}^m, l \leq m\}$ .

For the operator representation, now we reexpress the commutation relation of the super Heisenberg algebra (3.8) with new generators  $X_i, Y_i, \theta_a^1, \theta_a^2$  and  $I$  by the following map

$$q_i \longrightarrow iX_i, p_i \longrightarrow iY_i, \xi_a \longrightarrow i\theta_a^1, \lambda_a \longrightarrow i\theta_a^2, r \longrightarrow -iI. \quad (3.11)$$

Then we have

$$[X_i, Y_j] = i\delta_{ij}I, \quad \{\theta_a^1, \theta_b^2\} = i\delta_{ab}I, \quad \text{others} = 0. \quad (3.12)$$

A representation for  $X_i, Y_i, \theta_a^1, \theta_a^2$ , and  $I$  for operators can be given by

$$\begin{aligned} (X_i f)(s, \eta) &= s_i f(s, \eta), & (Y_i f)(s, \eta) &= -i \frac{\partial f}{\partial s_i}(s, \eta), \\ (\theta_a^1 f)(s, \eta) &= \eta_a f(s, \eta), & (\theta_a^2 f)(s, \eta) &= i \frac{\partial f}{\partial \eta_a}(s, \eta), \\ I &= \text{identity}, \end{aligned} \quad (3.13)$$

where  $s = (s_1, \dots, s_n), \eta = (\eta_1, \dots, \eta_m)$ , and  $(s, \eta)$  belongs to  $\mathbb{R}^{n|m}$ . Note that here again  $X_i, Y_i, \theta_a^1, \theta_a^2$  can be regarded as the coordinates of a non(anti)commutative superspace.

For the first operator representation, we send  $(t, x, y, \alpha, \beta)$  to  $\chi(t)\varepsilon(x, \alpha)\delta(y, \beta)$ , where  $(x, \alpha), (y, \beta) \in \mathbb{R}^{n|m}$ :

$$\begin{aligned} \chi(t) : f(s, \eta) &\longrightarrow e^{-it} f(s, \eta), \\ \varepsilon(x, \alpha) : f(s, \eta) &\longrightarrow e^{i(x \cdot X + \alpha \cdot \theta^1)} f(s, \eta) = e^{i(x \cdot s + \alpha \cdot \eta)} f(s, \eta), \\ \delta(y, \beta) : f(s, \eta) &\longrightarrow e^{i(y \cdot Y + \beta \cdot \theta^2)} f(s, \eta) = f(s + y, \eta + \beta). \end{aligned} \quad (3.14)$$

For the second operator representation, we send  $(t, x, y, \alpha, \beta)$  to  $U(t, x, y, \alpha, \beta) = \chi(t + \frac{xy + \alpha\beta}{2})\varepsilon(x, \alpha)\delta(y, \beta)$ . As in the bosonic case,  $U(t, x, y, \alpha, \beta)$  can be rewritten as

$$U(t, x, y, \alpha, \beta) = \exp[i(x \cdot X + y \cdot Y + \alpha \cdot \theta^1 + \beta \cdot \theta^2 - tI)]. \quad (3.15)$$

From now on, for the sake of brevity we will drop  $I$  for the identity, and will use the above form (3.15) in the following section.

Supersymmetric extensions of the Stone-von Neumann theorem were considered in [32, 21]. It was shown in [21] that there exists a unique irreducible unitary  $sH(2n|2m)$  module up to isomorphism described as in the bosonic case.

## 4. Deformed superspace

### 4.1 Deformed $\mathcal{N} = (2, 2)$ superspace in two dimensions

First we introduce the two-dimensional  $\mathcal{N} = (2, 2)$  superspace spanned by  $(X^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ .  $\theta$  and  $\bar{\theta}$  are transformed as a spinor and a conjugate spinor, respectively. We deform this superspace by the following commutation relations [10, 11]:

$$\begin{aligned} [X^1, X^2] &= i\Theta - 2iC\bar{\theta}^1\bar{\theta}^2, & [X^1, \theta^1] &= iC\bar{\theta}^2, & [X^1, \theta^2] &= iC\bar{\theta}^1, \\ [X^2, \theta^1] &= C\bar{\theta}^2, & [X^2, \theta^2] &= C\bar{\theta}^1, & \{\theta^1, \theta^2\} &= C, \end{aligned} \quad (4.1)$$

where  $\bar{\theta}$ 's commute or anticommute with other coordinates, i.e. the center.  $\Theta$  and  $C$  are constants. In the case of  $\Theta = 0$ , (4.1) is obtained by dimensional reduction of the commutation relation of non(anti)commutative  $\mathcal{N} = 1$  superspace in four dimensions.<sup>4</sup> On the other hand, in the case of  $C = 0$ , (4.1) reproduces the two-dimensional noncommutative space (with usual fermionic coordinates). Although we do not derive (4.1) from superstring, we use (4.1) as a simple unified expression of above two cases. The Moyal product corresponding to (4.1) is given by

$$\begin{aligned} F(X, \theta, \bar{\theta}) * G(X, \theta, \bar{\theta}) &= \\ \exp\left[\frac{i}{2}\Theta(\partial_1\partial'_2 - \partial_2\partial'_1) - \frac{1}{2}C(\mathcal{Q}_1\mathcal{Q}'_2 + \mathcal{Q}_2\mathcal{Q}'_1)\right] &F(X, \theta, \bar{\theta})G(X', \theta', \bar{\theta}') \Big|_{(X'^\mu, \theta', \bar{\theta}') = (X^\mu, \theta, \bar{\theta})}, \end{aligned} \quad (4.2)$$

where  $\partial_\mu = \partial/\partial X^\mu$ .  $\mathcal{Q}_1, \mathcal{Q}_2$  are the supercharges defined by

$$\mathcal{Q}_1 = \frac{\partial}{\partial\theta^1} - i\bar{\theta}^1(\partial_1 + i\partial_2), \quad \mathcal{Q}_2 = \frac{\partial}{\partial\theta^2} - i\bar{\theta}^2(\partial_1 - i\partial_2). \quad (4.3)$$

$\partial'_\mu, \mathcal{Q}'_1, \mathcal{Q}'_2$  are respectively obtained from  $\partial_\mu, \mathcal{Q}_1, \mathcal{Q}_2$  by replacement  $(X^\mu, \theta, \bar{\theta}) \rightarrow (X'^\mu, \theta', \bar{\theta}')$ . In general, from the consistency with supersymmetry, superspace can be deformed in terms of Moyal product which includes either supercharges  $\mathcal{Q}_\alpha$  or supercovariant derivatives  $D_\alpha$  in order to obtain non(anti)commutativity in fermionic coordinates, where  $D_\alpha$  is defined by

$$D_1 = \frac{\partial}{\partial\theta^1} + i\bar{\theta}^1(\partial_1 + i\partial_2), \quad D_2 = \frac{\partial}{\partial\theta^2} + i\bar{\theta}^2(\partial_1 - i\partial_2). \quad (4.4)$$

Now  $\mathcal{Q}_\alpha$  is used in the Moyal product as in (4.2). In this case, the half of the supersymmetry is broken but the chirality of superfields is preserved. If  $D_\alpha$  is used in the Moyal product instead of  $\mathcal{Q}_\alpha$ , the supersymmetry is fully preserved but the chirality of superfields is broken [7, 10].

Now we change the normalization of the operators by

$$X^\mu \rightarrow \sqrt{\Theta} X^\mu, \quad \theta^\alpha \rightarrow \sqrt{iC} \theta^\alpha, \quad \bar{\theta}^{\dot{\alpha}} \rightarrow \sqrt{\frac{\Theta}{iC}} \bar{\theta}^{\dot{\alpha}}. \quad (4.5)$$

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<sup>4</sup>In four dimensions, there are three non(anti)commutative parameters  $C^{11}, C^{12}$  and  $C^{22}$  in general. The parameter  $C$  in (4.1) corresponds to  $C^{12}$ . We have set  $C^{11} = C^{22} = 0$  for simplicity.

Then the commutation relations (4.1) become

$$\begin{aligned} [X^1, X^2] &= i - 2\bar{\theta}^1\bar{\theta}^2, & [X^1, \theta^1] &= \bar{\theta}^2, & [X^1, \theta^2] &= \bar{\theta}^1, \\ [X^2, \theta^1] &= -i\bar{\theta}^2, & [X^2, \theta^2] &= -i\bar{\theta}^1, & \{\theta^1, \theta^2\} &= -i, \end{aligned} \quad (4.6)$$

As in the bosonic case, we can introduce a unique operator representation of the above algebra as in (3.15), as a representative of the corresponding super Heisenberg group

$$U_A = \exp \left[ i \left( x_A X^1 + y_A X^2 + \alpha_A \theta^1 + \beta_A \theta^2 - \tilde{t}_A \right) \right]. \quad (4.7)$$

This satisfies

$$U_A U_B = \exp(-i \Xi_{AB}) U_B U_A, \quad (4.8)$$

$$\begin{aligned} \Xi_{AB} &= (1 + 2i\bar{\theta}^1\bar{\theta}^2)(x_A y_B - x_B y_A) + (\alpha_A \beta_B - \alpha_B \beta_A) \\ &\quad + i(x_A - i y_A) \beta_B \bar{\theta}^1 - i(x_B - i y_B) \beta_A \bar{\theta}^1 \\ &\quad - i(x_A + i y_A) \alpha_B \bar{\theta}^2 + i(x_B + i y_B) \alpha_A \bar{\theta}^2. \end{aligned} \quad (4.9)$$

Now, we can give two types of corresponding supermatrix representations. One is the  $4 \times 4$  supermatrix representation as

$$\pi_A = \exp \left[ M(\tilde{t}_A, x_A, y_A, \alpha_A, \beta_A) \right], \quad (4.10)$$

$$M(\tilde{t}_A, x_A, y_A; \alpha_A, \beta_A) = \left( \begin{array}{ccc|c} 0 & x_A & \tilde{t}_A & \tilde{\alpha}_A \\ 0 & 0 & y_A & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\beta}_A & 0 \end{array} \right), \quad (4.11)$$

where

$$\tilde{\alpha}_A = \alpha_A - i(x_A - i y_A) \bar{\theta}^1, \quad \tilde{\beta}_A = \beta_A + i(x_A + i y_A) \bar{\theta}^2. \quad (4.12)$$

In order to obtain (4.11), It is convenient to use the chiral coordinate  $Y^\mu$  which is defined by

$$Y^1 = X^1 + i\theta^1\bar{\theta}^1 - i\theta^2\bar{\theta}^2, \quad Y^2 = X^2 + \theta^1\bar{\theta}^1 + \theta^2\bar{\theta}^2. \quad (4.13)$$

The commutation relation of  $Y^\mu$  and  $\theta^\alpha$  is

$$[Y^1, Y^2] = i, \quad [Y^\mu, \theta^\alpha] = 0, \quad \{\theta^1, \theta^2\} = -i. \quad (4.14)$$

In terms of  $Y^\mu$ , (4.7) is rewritten as

$$U_A = \exp \left[ i \left( x_A Y^1 + y_A Y^2 + \tilde{\alpha}_A \theta^1 + \tilde{\beta}_A \theta^2 - \tilde{t}_A \right) \right]. \quad (4.15)$$

From (4.14) and (4.15), we obtain the representation (4.11). In this representation, we regard  $\bar{\theta}^1$  and  $\bar{\theta}^2$  as grassmann numbers.

For the other type of matrix representation, we regard  $\bar{\theta}^1$  and  $\bar{\theta}^2$  as the operators which belong to the center of the super Heisenberg group. The central element  $\tilde{t}_A$  in (4.7) is replaced with

$$\tilde{t}_A \rightarrow t_A + \bar{\alpha}_A \bar{\theta}^1 + \bar{\beta}_A \bar{\theta}^2 + u_A \bar{\theta}^1 \bar{\theta}^2, \quad (4.16)$$

where  $t_A, \bar{\alpha}_A, \bar{\beta}_A, u_A$  are the parameters corresponding to the center of the super Heisenberg group  $I, \bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^1\bar{\theta}^2$ , respectively. In this case, the representation is given by the  $6 \times 6$  supermatrix  $\tilde{\pi}_A$  as

$$\tilde{\pi}_A = \exp[\tilde{M}(t_A, x_A, y_A, u_A, \alpha_A, \beta_A, \bar{\alpha}_A, \bar{\beta}_A)], \quad (4.17)$$

$$\tilde{M}(t_A, x_A, y_A, u_A, \alpha_A, \beta_A, \bar{\alpha}_A, \bar{\beta}_A) = \left( \begin{array}{ccc|ccc} 0 & x_A & t_A & \alpha_A & 0 & \bar{\beta}_A \\ 0 & 0 & y_A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \beta_A & 0 & 0 & -i(x_A + iy_A) \\ 0 & 0 & \bar{\alpha}_A & i(x_A - iy_A) & 0 & u_A \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (4.18)$$

In terms of  $\tilde{\pi}_A$ , as in (3.1), we have the multiplication rule

$$(t_A, x_A, y_A, u_A, \alpha_A, \beta_A, \bar{\alpha}_A, \bar{\beta}_A) \cdot (t_B, x_B, y_B, u_B, \alpha_B, \beta_B, \bar{\alpha}_B, \bar{\beta}_B) = (t_{AB}, x_A + x_B, y_A + y_B, u_{AB}, \alpha_A + \alpha_B, \beta_A + \beta_B, \bar{\alpha}_{AB}, \bar{\beta}_{AB}). \quad (4.19)$$

Here  $t_{AB}, u_{AB}, \bar{\alpha}_{AB}$  and  $\bar{\beta}_{AB}$  are given by

$$\begin{aligned} t_{AB} &= t_A + t_B + \frac{1}{2}(x_A y_B - x_B y_A) + \frac{1}{2}(\alpha_A \beta_B - \alpha_B \beta_A), \\ u_{AB} &= u_A + u_B + i(x_A y_B - x_B y_A), \\ \bar{\alpha}_{AB} &= \bar{\alpha}_A + \bar{\alpha}_B + \frac{i}{2}(x_A - iy_A)\beta_B - \frac{i}{2}(x_B - iy_B)\beta_A, \\ \bar{\beta}_{AB} &= \bar{\beta}_A + \bar{\beta}_B - \frac{i}{2}(x_A + iy_A)\alpha_B + \frac{i}{2}(x_B + iy_B)\alpha_A. \end{aligned} \quad (4.20)$$

The commutation relation among  $\tilde{\pi}$ 's is the same as that of  $U$ 's in (4.8):

$$\tilde{\pi}_A \tilde{\pi}_B = \exp(\Omega_{AB}) \tilde{\pi}_B \tilde{\pi}_A, \quad (4.21)$$

where

$$\Omega_{AB} = \left( \begin{array}{ccc|ccc} 0 & 0 & t_{[AB]} & 0 & 0 & \bar{\beta}_{[AB]} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\alpha}_{[AB]} & 0 & 0 & u_{[AB]} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (4.22)$$

and the bracket means antisymmetrization of indices.

## 4.2 Toy superspace with non-central extension

Super Heisenberg group is the central extension of the ordinary superspace. However, some of noncommutative superspaces which cannot be obtained by central extension admit the matrix representation similar to super Heisenberg group. In this section, we consider such an example of the superspace with non-central extension.

Here we start from the module which is given by  $f(s, \zeta, \bar{\zeta})$ , where  $(s, \zeta, \bar{\zeta})$  is the coordinates of ordinary one-dimensional superspace. From this module, we define the coordinates of the two-dimensional deformed superspace by<sup>5</sup>

$$\hat{X}^1 = s \cdot, \quad \hat{X}^2 = -i \frac{\partial}{\partial s}, \quad \hat{\theta}^1 = -i\zeta \cdot, \quad \hat{\theta}^1 = -i\bar{\zeta} \cdot, \quad \hat{\theta}^2 = \frac{\partial}{\partial \zeta} + \bar{\zeta} \frac{\partial}{\partial s}, \quad \hat{\theta}^2 = \frac{\partial}{\partial \bar{\zeta}} + \zeta \frac{\partial}{\partial s}. \quad (4.23)$$

The nontrivial commutation relations are

$$\begin{aligned} [\hat{X}^1, \hat{X}^2] &= i, & [\hat{X}^1, \hat{\theta}^2] &= -i\hat{\theta}^1, & [\hat{X}^1, \hat{\theta}^1] &= -i\hat{\theta}^1, \\ \{\hat{\theta}^1, \hat{\theta}^2\} &= -i, & \{\hat{\theta}^1, \hat{\theta}^1\} &= -i, & \{\hat{\theta}^2, \hat{\theta}^2\} &= 2i\hat{X}^2. \end{aligned} \quad (4.24)$$

The last equation in (4.24) resembles the deformed superspace defined in [9]. Now the commutators and the anticommutators contain the operators which are not centers. Then this superspace is not a central extension of the ordinary two-dimensional superspace.<sup>6</sup> Despite this, the matrix representation of (4.24) can be constructed as in the case of super Heisenberg group. We define  $\hat{U}_A$  as follows like the representative of super Heisenberg group in the previous subsection

$$\hat{U}_A = \exp \left[ i \left( \hat{x}_A \hat{X}^1 + \hat{y}_A \hat{X}^2 + \hat{\alpha}_A \hat{\theta}^1 + \hat{\beta}_A \hat{\theta}^2 + \hat{\alpha}_A \hat{\theta}^1 + \hat{\beta}_A \hat{\theta}^2 - \hat{t}_A \right) \right]. \quad (4.25)$$

This satisfies

$$\hat{U}_A \hat{U}_B = \exp(-i \hat{\Xi}_{AB}) \hat{U}_B \hat{U}_A, \quad (4.26)$$

where  $\hat{\Xi}_{AB}$  is given by

$$\begin{aligned} \hat{\Xi}_{AB} &= (\hat{x}_A \hat{y}_B - \hat{y}_A \hat{x}_B) + (\hat{\alpha}_A \hat{\beta}_B + \hat{\beta}_A \hat{\alpha}_B) + (\hat{\alpha}_A \hat{\beta}_B + \hat{\beta}_A \hat{\alpha}_B) \\ &+ \frac{3}{2} (\hat{\beta}_A \hat{\beta}_B + \hat{\beta}_A \hat{\beta}_B) (\hat{x}_A + \hat{x}_B) - (\hat{x}_A \hat{\beta}_B - \hat{\beta}_A \hat{x}_B) \hat{\theta}^1 \\ &- (\hat{x}_A \hat{\beta}_B - \hat{\beta}_A \hat{x}_B) \hat{\theta}^1 - 2(\hat{\beta}_A \hat{\beta}_B + \hat{\beta}_A \hat{\beta}_B) \hat{X}^2. \end{aligned} \quad (4.27)$$

We can assign the corresponding matrix representation of  $\hat{U}_A$  as the supermatrix  $\hat{\pi}_A$

$$\hat{\pi}_A = \exp \left[ \hat{M} \left( \hat{t}_A, \hat{x}_A, \hat{y}_A, \hat{\alpha}_A, \hat{\alpha}_A, \hat{\beta}_A, \hat{\beta}_A \right) \right], \quad (4.28)$$

where  $\hat{M}$  is a  $5 \times 5$  supermatrix given by

$$\hat{M} \left( \hat{t}_A, \hat{x}_A, \hat{y}_A, \hat{\alpha}_A, \hat{\alpha}_A, \hat{\beta}_A, \hat{\beta}_A \right) = \left( \begin{array}{ccc|cc} 0 & \hat{x}_A & \hat{t}_A & \hat{\alpha}_A & \hat{\alpha}_A \\ 0 & 0 & \hat{y}_A & \hat{\beta}_A & \hat{\beta}_A \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \hat{\beta}_A & 0 & 0 \\ 0 & 0 & \hat{\beta}_A & 0 & 0 \end{array} \right). \quad (4.29)$$

<sup>5</sup>Here we have already changed the normalization of the operators to absorb noncommutative and non-anticommutative parameters as in the case of (4.6).

<sup>6</sup>However the bosonic part of this algebra is Heisenberg algebra.

It is straightforward to check that  $\hat{\pi}$  satisfies the same commutation relation (4.26) of  $\hat{U}$ :

$$\hat{\pi}_A \hat{\pi}_B = \exp(\hat{\Omega}_{AB}) \hat{\pi}_B \hat{\pi}_A, \quad (4.30)$$

where

$$\hat{\Omega}_{AB} = \left( \begin{array}{cc|cc} 0 & 0 & \hat{t}_{[AB]} & \hat{\alpha}_{[AB]} \\ 0 & 0 & \hat{y}_{[AB]} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (4.31)$$

with  $\hat{t}_{[AB]}$ ,  $\hat{\alpha}_{[AB]}$ ,  $\hat{\hat{\alpha}}_{[AB]}$ ,  $\hat{y}_{[AB]}$  given as follows:

$$\begin{aligned} \hat{t}_{[AB]} &= (\hat{x}_A \hat{y}_B - \hat{y}_A \hat{x}_B) + (\hat{\alpha}_A \hat{\beta}_B + \hat{\beta}_A \hat{\alpha}_B) + (\hat{\hat{\alpha}}_A \hat{\hat{\beta}}_B + \hat{\hat{\beta}}_A \hat{\hat{\alpha}}_B) \\ &\quad + \frac{3}{2} (\hat{\beta}_A \hat{\hat{\beta}}_B + \hat{\hat{\beta}}_A \hat{\beta}_B) (\hat{x}_A + \hat{x}_B), \\ \hat{\alpha}_{[AB]} &= \hat{x}_A \hat{\hat{\beta}}_B - \hat{\hat{\beta}}_A \hat{x}_B, \\ \hat{\hat{\alpha}}_{[AB]} &= \hat{x}_A \hat{\beta}_B - \hat{\beta}_A \hat{x}_B, \\ \hat{y}_{[AB]} &= 2(\hat{\beta}_A \hat{\hat{\beta}}_B + \hat{\hat{\beta}}_A \hat{\beta}_B). \end{aligned} \quad (4.32)$$

This confirms that we can use the above matrix representation instead of dealing with more complicated operator manipulations.

## 5. Conclusion

In this paper, we construct the super Heisenberg group and corresponding supermatrix representation by extending the result known in the bosonic case.

The low-energy effective theory of D-branes in a background NSNS  $B$ -field becomes the noncommutative field theory, and the algebra of the coordinates becomes Heisenberg algebra. From Heisenberg algebra, Heisenberg group can be constructed by exponential mapping. The matrix representation of Heisenberg group is useful to construct noncommutative tori and quantum theta-functions. One can construct noncommutative tori in a easier manner via the embedding of the corresponding Heisenberg groups.

When the background RR field is turned on, the low-energy effective theory of D-branes becomes the field theory on non(anti)commutative superspace of which the fermionic coordinates have nontrivial commutation relations, and super Heisenberg group would appear. For the analogue of bosonic Heisenberg group, the supermatrix representation of super Heisenberg group can be constructed. We explicitly carry out this construction by extending the relation known in the bosonic case to the super case: the two-dimensional deformed  $\mathcal{N} = (2, 2)$  superspace containing non(anti)commutativity in both bosonic and fermionic coordinates. As in the bosonic case, the supermatrix representation of super Heisenberg group would be useful to construct noncommutative supertori and quantum super theta-functions [31].

Furthermore, this supermatrix representation is also applicable to deformed superspaces corresponding to non-centrally extended ‘super Heisenberg groups’. We demonstrate this construction with a toy model of two-dimensional deformed superspace at the end.

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